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Abelian sandpile model

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A systematic and simple method to find the correlation function of the Abelian sandpile model up to any finite order is developed. In addition, an algorithm for evaluating the distribution function of the avalanche size $P(s)$ exactly is also discovered along the same line. This method is, in general, more efficient (and accurate) than the numerical simulation that is currently used in obtaining $P(s)$.

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Much work has been done to understand the nature of temporal and spatial scaling in some physical systems without fine-tuning parameters since the introduction of the concept of self-organized criticality by Bak, Tang, and Wiesenfeld [1]. Later on, Dhar pointed out that the finite cellular-automata type of model with toppling triggered by local height can be described by a finite Abelian group [2]. Models of this type are now collectively known as the Abelian sandpile models (ASM's). Both the total number of self-organized critical states and the two-point correlation function can be calculated exactly in these models [2,3]. Furthermore, various other physical quantities of ASM's are found [4]. Extension of the ASM, whose toppling is triggered by local height, to other triggering conditions of toppling can be found elsewhere [5,6]. Recently, the relation between the ASM, percolation, and spanning trees is also explored [7].

In this Rapid Communication, I would like to introduce an exact and efficient algorithm for calculating the correlation function up to any finite order for the ASM. Thus the distribution function of avalanche sizes can be found exactly in this model.

An Abelian sandpile model consists of a finite number of cells (labeled i), and integers (or sometimes real numbers) called local heights (labeled h_i) are assigned to each of the cells. We denote the total number of cells by N in the forthcoming discussions. A unit amount of particles is added to the cells randomly, according to a prescribed probability distribution $\mu(i)$. Whenever the local height of a cell j is greater than a fixed triggering level, particles have to be redistributed in the next time step according to the rule (called the toppling rule),

$$h_i \longrightarrow h_i - \Delta_{ji} \quad \forall\, i. \tag{1}$$

This process is called toppling. The requirements and properties of the toppling matrix Δ can be found elsewhere [2,3]. As pointed out by Dhar, the total number of self-organized critical states in an ASM is given by $\det \Delta$. In the event that the local heights h_i are real numbers, $\det \Delta$ can be interpreted as the volume of the phase space of the recurrence states [5,6]. In addition, the two-point correlation function G_{ij}, which is the average number of topplings occuring in the cell j due to a particle added onto cell i in an avalanche, is given by $G_{ij} = \Delta^{-1}_{ij}$ [2]. One of the goals of this paper is to extend this result up to any finite number of cells involved.

Let us denote the average number of toppling at cell j, given that a particle is added onto cell i and that cell k does not topple during the avalanche, by $G_{ij|\bar{k}}$. Then it is given by

$$G_{ij|\bar{k}} = \frac{[\mathrm{Adj}\Delta_{(k)}]_{ij}}{\det \Delta} \tag{2}$$

whenever $k \neq i$, and $k \neq j$. Here $\Delta_{(k)}$ denotes the $(N-1) \times (N-1)$ matrix formed by removing the column k and row k of Δ, and Adj denotes the adjoint of a matrix.

The proof of the above assertion follows directly from the following observation: a digraph, called the induced graph, is formed by taking each cell as a vertex, and an arrow is drawn from vertex i to another vertex $j(\neq i)$, provided that $\Delta_{ij} \neq 0$ [6]. Let us consider a new digraph by removing the vertex k and all the edges pointing to or away of k. The toppling matrix associated with this new graph is therefore $\Delta_{(k)}$. Clearly, there is a one-to-one correspondence between the avalanches occurring in the new system (with $\Delta_{(k)}$) and the old one (with Δ), in which k is not involved and hence not toppled. However, the total number of self-organized critical states in the new system is $\det \Delta_{(k)}$ instead of $\det \Delta$. Thus, $G_{ij|\bar{k}} = \det \Delta_{(k)} \Delta^{-1}_{(k)\ ij} / \det \Delta$ and so Eq. (2) is true.

By removing any finite number of rows and the corresponding columns in the toppling matrix Δ, we can calculate the corresponding higher-order correlation functions for the rest of the cells in a similar way. Moreover, all the correlation functions up to any finite order of the ASM can be found by means of a similar iteration. For example, the three- and four-point correlation functions can be found using the following relations in conjunction with Eq. (2):

$$G_{ij|k} = G_{ij} - G_{ij|\bar{k}}, \tag{3a}$$

$$G_{ij|\bar{k}l} = G_{ij|\bar{k}} - G_{ij|\bar{k}\bar{l}}, \tag{3b}$$

and

$$G_{ij|kl} = G_{ij|k} - G_{ij|k\bar{l}}, \tag{3c}$$

where $G_{ij|k}$ is the average number of toppling occurring in cell j, given that a particle is added in cell i and cell k topples, and similarly for $G_{ij|kl}$ and $G_{ij|k\bar{l}}$.

Show the probability of cell j toppling, given that a particle is added onto cell i by $P_i(j)$. In general, $P_i(j) \leq G_{ij}$, because a cell may topple more than once in an avalanche, and hence the equality holds if and only if the cell j topples exactly once in every avalanche involving it. Let us consider another Abelian sandpile system with the toppling matrix $\tilde{\Delta}$ given by

$$\tilde{\Delta}_{pq} = \begin{cases} 0 & \text{if } p = j \text{ and } q \neq j \\ \Delta_{pq} & \text{otherwise,} \end{cases} \tag{4}$$

that is, we remove all the out-going edges from the cell j in the induced graph. By simple counting, it is easy to see that there is a one-to-one correspondence between the self-organized critical states of these two systems, to which cell j does not topple when a particle is added to cell i. Thus the total number of self-organized critical states with cell j not toppled, given a particle is added to cell i, is the same for the two systems (with toppling rules given by Δ and $\tilde{\Delta}$, respectively) above. So the probabilities $P_i(j)$ and $\tilde{P}_i(j)$ of these two systems are inversely proportional to their corresponding numbers of self-organized critical states. In addition, the probability $P_i(j)$ is unchanged upon elementary row operations on Δ not involving the row vector representing the toppling rule of cell j. The reason is simple: the number of self-organized critical states (or the recurrence phase-space volume) is unchanged by elementary row operations. In addition, those operations not involving cell j can only affect the mode of transient particle transport of cells not involving j. So the particle eventually coming into cell j via toppling remains unchanged. So our claim is justified. Further discussions of the role of elementary row operations on the ASM and the equivalence of toppling rules can be found elsewhere [2,8]. Using the above procedure, which is similar to the Gaussian elimination, we can always reduce the toppling matrix $\tilde{\Delta}$ uniquely to a new one Δ' with the only (possible) nonzero elements being those in the diagonal and in the column vector corresponding to the strength of the edges pointing to j in the induced graph of Δ'. Hence the induced graph of Δ' is a forest. If $-\Delta'_{ij} > \Delta'_{jj}$, it takes, in general, more than one toppling in j in order that j can release all the excess particles that have moved into it due to the instability in i. However, if we define Δ''_{ij} by

$$\Delta''_{ij} = \max\left(\Delta'_{ij}, -\Delta'_{jj}\right), \tag{5}$$

then it is easy to check that cell j can topple at most once in an avalanche. Thus $G''_{ij} = (\Delta''^{-1})_{ij} = P''_i(j)$ and hence $P_i(j)$ is given by

$$P_i(j) = \frac{\det \Delta''}{\det \Delta}\, G''_{ij} = \frac{(\mathrm{Adj}\Delta'')_{ij}}{\det \Delta}. \tag{6}$$

So $P_i(\bar{j}) = 1 - P_i(j)$, the probability that j does not topple, given a particle is added to cell i, can be found. Inductively, we can apply a similar procedure to find $P_i(\bar{j}_1, \ldots, \bar{j}_m)$ for any distinct $j_1, \ldots, j_m$ and is given by

$$P_i(\bar{j}_1, \ldots, \bar{j}_m) = 1 - \frac{1}{\det \Delta} \sum_{p=1}^{m} \left[(\mathrm{Adj}\tilde{\Delta})_{ij_p} \prod_{l=1}^{p-1} \left(1 - \tilde{\Delta}^{-1}_{ij_l}\right) \right], \tag{7}$$

where $\tilde{\Delta}$ is obtained by first removing all the outgoing edges of j_p and then transforms by means of elementary row operations to the form where all $\tilde{\Delta}_{ik} = 0$ whenever $k \notin \{j_1, \ldots, j_m\}$. Finally, the elements in $\tilde{\Delta}_{ij_p}$ are checked to comply with the constraint in the form of Eq. (5). In fact, the product in the second term of Eq. (7) is the probability that the cells $j_1, j_2, \ldots, j_{p-1}$ do not topple under the toppling rules of $\tilde{\Delta}$, and so using the same argument it is not difficult to see that the second term of Eq. (7) is the probability that at least one of the cells $j_1, j_2, \ldots, j_m$ topples when a particle is added to cell i, with the toppling matrix given by Δ.

With the aid of the $P_i(\bar{j}_1, \ldots, \bar{j}_k)$, all the probability functions up to any finite order of the ASM can be found by means of iteration. For example,

$$P_i(j) = 1 - P_i(\bar{j}), \tag{8a}$$

$$P_i(j\bar{k}) = P_i(\bar{k}) - P_i(\bar{j}\bar{k}), \tag{8b}$$

$$P_i(jk) = P_i(j) - P_i(j\bar{k}). \tag{8c}$$

Let us define an avalanche size as the total number of cells that topple during an avalanche, and those that topple more than once only count once. Then it is straightforward to show that the probability of having an avalanche of size s, given that a particle is added to cell i, $\mathcal{P}_i(s)$, is given by

$$\mathcal{Q}_i(N-k) = \sum_{q=0}^{k} {}^{N-q}C_{k-q}\, \mathcal{P}_i(q) \tag{9}$$

for $k = 0, 1, \ldots, N$, where N is the total number of cells of the ASM, and $\mathcal{Q}_i(k)$ is given by

$$\mathcal{Q}_i(k) = \sum P_i(\bar{j}_1, \ldots, \bar{j}_k), \tag{10}$$

where the sum is over all the distinct combinations of any k of the N cells. Actually Eq. (9) follows from the simple combinatorial argument as follows: clearly avalanches involving more than k cells do not contribute to Eq. (10). Moreover, for an avalanche involving q cells ($q = 0, 1, \ldots, k$), the sum in Eq. (10) counts this event ${}^{N-q}C_{k-q}$ times. So Eq. (9) is valid. In fact, it can be regarded as a matrix equation, so that $\mathcal{P}_i(m)$ can be solved by backward substitution. If $\mu(i)$ denotes the probability of adding a particle to cell i, the probability of having an avalanche of size m is given by $\mathcal{P}(m) = \sum_i \mu_i \mathcal{P}_i(m)$.

Approximately N arithmetic operations are involved in finding $\mathcal{P}_i(k)$ from $\mathcal{Q}_i(N-k)$, $\sim {}^{N}C_k$ operations in finding $\mathcal{Q}_i(k)$ from its summand $P_i(\bar{j}_1, \ldots, \bar{j}_k)$, and finally $\sim k^4$ operations in finding $P_i(\bar{j}_1, \ldots, \bar{j}_k)$ from the toppling matrix Δ. Therefore the total operation count of the finding of $\mathcal{P}(m) = \sum_i \mu(i)\mathcal{P}_i(m)$ scales as

$N^2 \sum^N C_k k^4 \approx N^6 2^N$, where N is the total number of cells. It is expected that the total number of recurrence states $\det \Delta$ of a system scale as ζ^N for some $\zeta > 0$ [7]. So the operation count used to obtain $\mathcal{P}(m)$ by direct computer simulation should scale as $Nt\zeta^N \sim N^2\zeta^N$, where t is the average time elapse for an avalanche. As soon as $\zeta > 2$, which is almost always the case, the algorithm we have proposed here will be more efficient than that of computer simulation.

Finally, it should be noted that there are redundant terms in the evaluation of $\mathcal{Q}_i(k)$ by means of Eq. (10). Clearly, the cells involved in an avalanche form a percolating animal of the induced graph (note, however, that the converse is not true in general). Therefore, we can accelerate the process by counting in Eq. (9) and by summing only over those configurations where the cells that are allowed to topple in an avalanche form a percolating animal in the induced graph. So Eq. (9) can be modified to

$$\tilde{\mathcal{Q}}_i(N-k) = g_i(k) \sum_{q=0}^{k} \frac{1}{g_i(q)} \mathcal{P}_i(q) \tag{11}$$

for $k = 0, 1, \ldots, N$, where $g(q)$ $(q > 1)$ is the number of percolating animals of the induced graph, starting from cell i with size q, $g_i(0) = 1$, and $\tilde{\mathcal{Q}}_i(k)$ is defined by

$$\tilde{\mathcal{Q}}_i(k) = \sum{}' P_i(\bar{j}_1, \ldots, \bar{j}_k), \tag{12}$$

where the prime sum is over all the combinations of $j_1, \ldots, j_k$, with the remaining cells forming a percolating animal starting from the cell i for the induced graph. It can be shown that $g(k)$ cannot grow faster than $\exp(\alpha k)$ for some $\alpha > 0$ as $k \to \infty$ [9]. If the induced graph is a subset of a regular finite-dimensional lattice, we expect that $g(k) \sim k^\alpha$ for some $\alpha > 1$; thus the total number of arithmetic operations involved scales as a power law of N, which is a very efficient way to calculate $P_i(k)$ and hence $P(k)$ either analytically or numerically without using computer simulation.

In this Rapid Communication, we have developed a systematic method to calculate all the correlation functions up to any finite order of any Abelian sandpile model. In a similar way, the conditional probability of any finite number of sites being (or equivalently not being) toppled is also found. An efficient algorithm for evaluating the distribution of avalanche size $P(s)$ is therefore discovered. Numerical work on a specific model is underway.

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